Dynamical Origins of Distribution Functions

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ABSTRACT

Many real-world problems are time-evolving in nature, such as the progression of diseases, the cascading process when a post is broadcasting in a social network, or the changing of climates. The observational data characterizing these complex problems are usually only available at discrete time stamps, this makes the existing research on analyzing these problems mostly based on a cross-sectional analysis. In this paper, we try to model these time-evolving phenomena by a dynamic system and the data sets observed at different time stamps are probability distribution functions generated by such a dynamic system. We propose a theorem which builds a mathematical relationship between a dynamical system modeled by differential equations and the distribution function (or survival function) of the cross-sectional states of this system. We then develop a survival analysis framework to learn the differential equations of a dynamical system from its cross-sectional states. With such a framework, we are able to capture the continuous-time dynamics of an evolutionary system. We validate our framework on both synthetic and real-world data sets. The experimental results show that our framework is able to discover and capture the generative dynamics of various data distributions accurately. Our study can potentially facilitate scientific discoveries of the unknown dynamics of complex systems in the real world.

CCS CONCEPTS

• **Mathematics of computing → Distribution functions**; **Ordinary differential equations**; **Survival analysis**;

KEYWORDS

Distribution Functions; ODE; Differential Equation Systems; Dynamical Modeling; Survival Analysis;

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1 INTRODUCTION

Most of the real world problems are dynamic in nature, however, what we can observe are only cross-sectional states at specific time points. Inferring the evolutionary dynamics of a complex system from the observations of its cross-sectional states is a fundamental research problem in various domains, ranging from biology [10], physics [3], social science [14], to computer science [21], etc. For example, Barabási and Albert [5] studied how complex networks grow over time by observing their cross-sectional power-law degree distributions. Yoshida *et al.* [22] inferred the evolutionary mechanisms of an ecological system from its cross-sectional abundance of species. Sinatra *et al.* [18] and Wang *et al.* [19] explored how scholars' scientific impact changes at long term by studying the citation distribution of their papers. Oliveira and Barabási [16] inferred the decision-making dynamics of human beings from the inter-event time distribution of the correspondence patterns of Darwin and Einstein (or from the online collaborations behaviors [28]). Pierson *et al.* [17] inferred the longitudinal progression patterns of diseases from cross-sectional patients' records. In those cases, it is usually difficult to infer the continuous-time evolutionary dynamics of the system because of its complexity and/or the lack of longitudinal records. Moreover, the existing research on understanding such dynamics is mostly case-by-case in different scenarios. To the best of our knowledge, there is no prior work trying to derive a general theoretical framework to reveal the intrinsic relationship between various dynamical systems and the distributions of their cross-sectional states. Such a relationship can further be helpful of learning the evolutionary dynamics of complex systems directly from their cross-sectional data samples. This can facilitate scientific discoveries of the unknown evolutionary mechanisms of complex dynamical systems (e.g., complex social, biological, environmental and environmental systems, etc.) in the real world, based on which we can make better prediction [11] and control [12].

We aim at filling in such a gap in this paper. Specifically, we theoretically prove that the distribution function (e.g., the heavy-tailed, narrow-tailed, power-law or S-shaped, etc.) of the cross-sectional states (e.g., nodes' degree, citation number, wealth, cities' or tumors' size, etc.) of a dynamical system, is generated by a dynamical system modeled by differential equation $\frac{dx_i(t)}{dt}|_{x_0} = \frac{dF^{-1}(1-\frac{t_i}{t})}{dt} = \frac{d\Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}t))}{dt}$ with uniform random signals as inputs (Theorem 2.1 and Corollary 2.1.1). Under this theorem, we prove the equivalence between probability distributions and their generative dynamics, and develop a set of statistical tools to infer their possible generative dynamics from the cross-sectional distributions of observed data samples.

In order to demonstrate the capabilities of our theory, in Table 1 we derive the generative dynamics of several distributions describing the cross-sectional states of dynamical systems, ranging from narrow-tailed distributions to heavy-tailed ones, and then find new cross-sectional distributions of several typical dynamical systems composed of interpretable mechanisms like preferential attachment, growth competition, environment limit, and so on [4]. Many of these derived distributions and dynamical systems are discovered for the first time. Furthermore, we propose a showcased statistical model based on survival analysis to encompass typical dynamical mechanisms, which enables the learning process of the underlying dynamics of complex systems directly from data.

We validate our framework on both synthetic and real-world data sets. As for the synthetic data sets, our framework recovers the ground-truth distributions (covering narrow-tailed, heavy-tailed and mixture distributions) and their generative dynamics accurately. We then apply our framework to a wide range of real-world data sets to infer their generative dynamics. Even if the distributions of real-world data sets exhibit more complexities, our framework is able to capture and reproduce the real-world data sets accurately. This implies that our approach can uncover plausible generative dynamics of the observed data. It is worthwhile to highlight our contributions as follows:

- **A Novel Theorem:** We give a theorem which builds a mathematical relationship between a dynamical system described in differential equations and the distribution or survival function of the cross-sectional states of this dynamical system. (Section 2).
- **Novel Findings:** By applying our theorem, we have discovered many new distributions, new hazard functions in survival analysis and new differential equations describing dynamical systems with interpretable mechanisms (Section 3, see Table 1).
- **A Practical Model:** We propose a statistical model to learn the differential equations of unknown dynamical systems directly from empirical datasets (Section 4).

2 PROPOSED THEOREM

2.1 Notations

Notations in probability theory, survival analysis, point process [1] and dynamical systems /differential equations [4] are used:

- *Probability theory:* Let X be a random variable generated from a cumulative distribution function $F(X \le x) = \int_{x_0}^{x} f(s)ds = 1 - S(X > x)$, and we observe n data samples $x_1, ..., x_n$. Here we assume the $F(X)$ is absolutely continuous. The $f(x)$ and $S(x) = 1 - F(x)$ are the probability density function and the survival function (or complementary cumulative distribution function) of the X respectively.
- *Survival analysis:* The hazard function $\lambda(x)$ of the X is defined as:

$$\lambda(x) = \lim_{\Delta x \to 0^+} \frac{Pr(x \le X < x + \Delta x | X \ge x)}{\Delta x} = \frac{f(x)}{S(x)}, \qquad (1)$$

  interpreted as the the probability of X sampled with value x^+ conditional on X not being sampled with value smaller than x. We define $\Lambda(x) = \int_{x_0}^{x} \lambda(s)ds$ as the cumulative hazard function. Due to the fact that $\Lambda(x)$ is monotonically increasing, thus $\Lambda(x)$ is invertable and we define $\Lambda^{-1} : \mathscr{R}^+ \to \mathscr{R}$, $\Lambda^{-1}(\Lambda(x)) = x$. Similarly, we can define $F^{-1} : \mathscr{R}^+ \to \mathscr{R}$, $F^{-1}(F(x)) = x$.
- *Point process:* We use $\mathscr{P}(t|\lambda_p) = \{t_1, ..., t_i, ...|0 < t_1 \le ... \le t_i \le .. \le t\}$ to denote a Poisson point process until time t with the occurrence time t_i of event i, and intensity rate $\lambda_p > 0$. Its equivalent counting process is $N(t|\lambda_p) = \sum_{i \ge 1} \mathbb{1}_{(0,t]}(t_i)$.
- *Dynamical system:* We define a dynamical system changing over time t by differential equations $\mathscr{D}(t) = \{x_i(t)| \frac{dx_i(t)}{dt}, x_i(t_i), i = 1, 2, ...\}$, where $x_i(t)$ is the state of the i^{th} agent who joined the system at time $t_i > 0$, and the state of agent i is changing according to the differential equation $\frac{dx_i(t)}{dt}$ with initial value $x_i(t_i)$.

2.2 Proposed Theorem and Corollary

Our main theorem gives the generative dynamics of an arbitrary distribution function $F(x)$ as follows:

Theorem 2.1. *Given a dynamical system $\mathscr{D}(t) = \{x_i(t) > 0| \frac{dx_i(t)}{dt}, x_i(t_i) = x_0; i = 1, 2, ...\}$ consisting of agent i who arrives in the system at time t_i according to a Poisson process $\mathscr{P}(t|\lambda_p) = \{t_1, ..., t_i, ...|0 < t_1 \le ... \le t_i \le ... \le t\}$, the state of agent i changes according to a differential equation $\frac{dx_i(t)}{dt}|_{x_0}$ with initial value x_0, and the cross-sectional states of $\mathscr{D}(t)$ at time point t, namely $x(t) = \{x_1(t), ..., x_i(t), ...\}$, follows the distribution $F(x(t))$ if and only if $\frac{dx_i(t)}{dt}|_{x_0} = \frac{dF^{-1}(1-\frac{t_i}{t})}{dt}$.*

Corollary 2.1.1. *Under the same conditions as Theorem 2.1, the cross-sectional states $x(t) = \{x_1(t), ..., x_i(t), ...\}$ of $\mathscr{D}(t)$ at time t follows distribution $F(x(t))$ if and only if $\frac{dx_i(t)}{dt}|_{x_0} = \frac{d\Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}t))}{dt}$.*

Usually, the survival function $S(x_0) = 1$ at the initial state x_0, and thus we get the generative dynamics as $\frac{dx_i(t)}{dt}|_{x_0} = \frac{d\Lambda^{-1}(\ln \frac{t}{t_i})}{dt}$.

According to the theorem and the corollary, the statistical properties of $F(x)$, say heavy tail, narrow tail, power law, or S-shape etc., essentially originate from the growth dynamics $\frac{dx_i(t)}{dt}|_{x_0} = \frac{dF^{-1}(1-\frac{t_i}{t})}{dt} = \frac{d\Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}t))}{dt}$ consisting of agents who randomly comes to the system at a constant speed with same initial state x_0.

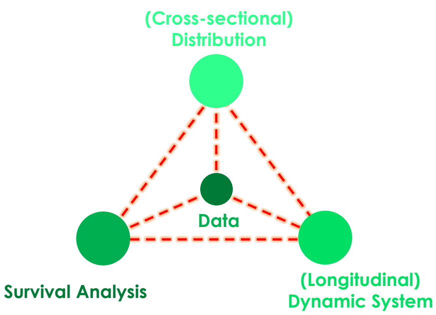


Figure 1: *Our theorem and the corollary connect the dots above.*

By applying our Theorem 2.1 and Corollary 2.1.1, we try to build a mathematical relationship between a dynamical system governed by differential equations, and the distributions or hazard functions of the system's cross-sectional states as shown in Fig. 1. Examples are shown in Table 1 and discussed in detail in Section 3.

2.3 Proof

Based on Lemma 2.2, 2.3 and 2.4, we show the detailed proof of Theorem 2.1 and the Corollary 2.1.1.

LEMMA 2.2. *[8] Given a Poisson process $\mathscr{P}(t|\lambda_p) = \{t_1, ..., t_i, ...|0 < t_1 \le ... \le t_i \le .. \le t\}$ with $N(t|\lambda_t) = n$, then the probability density function of a random event time t_i given the total time t is $f(t_i) = \frac{1}{t}$, indicating a uniform distribution on $(0, t]$.*

PROOF. The joint probability density function of random variables $t_i, i = 1, ..., n$ is:

$$\begin{aligned} &Pr(t_i < T_i \le t_i + \delta_i, i = 1, \ldots, n | N(t) = n) = \\ &\left[\frac{\begin{array}{r} Pr(N(t_i + \delta_i) - N(t_i) = 1, N(t_{j+1}) - N(t_j + \delta_j) = 0, \\ i = 1, \ldots, n, j = 0, \ldots, n, t_0 = 0, \delta_0 = 0) \end{array}}{Pr(N(t) = n)} \right] \\ &= \frac{\prod_{i=1}^n \lambda_p \delta_i e^{-\lambda_p \delta_i} e^{-\lambda_p (t - \sum_{i=1}^n \delta_i)}}{e^{-\lambda_p t} (\lambda_p t)^n / n!} = \frac{n!}{t^n} \prod_{i=1}^n \delta_i, \end{aligned} \tag{2}$$

and thus:

$$\begin{aligned} &f(t_i, i = 1, \ldots, n | N(t) = n) \\ &= \lim_{t_i \to 0, i=1,\ldots,n} \frac{Pr(t_i < T_i \le t_i + \delta_i, i = 1, \ldots, n | N(t) = n)}{\prod_{i=1}^n \delta_i} \\ &= \frac{n!}{t^n}, \end{aligned} \tag{3}$$

where $0 < t_1 \le ... \le ti \le ... \le t$. For the order statistics $t_i, i = 1, ..., n$, $f(t_i) = \frac{1}{t}$. □

LEMMA 2.3. *[9] Given a random variable u which follows uniform distribution $U(0, 1]$, then $x = F^{-1}(u)$ follows distribution $F(x)$. Also, if X follows distribution $F(x)$, then $F(x)$ follows uniform distribution $U(0, 1]$.*

PROOF. The cumulative density function of random variables x is:

$$\begin{aligned} &Pr(F^{-1}(u) \le x) = Pr(u \le F(x)) = F(x), \text{ and also} \\ &Pr(F(x) \le u) = Pr(x \le F^{-1}(u)) = F(F^{-1}(u)) = u \end{aligned} \tag{4}$$

□

LEMMA 2.4. *Given a random variable u which follows uniform distribution $U(0, 1]$, then $x = \Lambda^{-1}(\ln \frac{S(x_0)}{u})$ follows distribution $F(x)$. Also, $S(x_0)e^{-\Lambda(x)}$ follows uniform distribution $U(0, 1]$.*

PROOF. According to the definition of hazard function $\lambda(x)$,

$$\lambda(x) = \frac{f(x)}{S(x)} = \frac{-S'(x)}{S(x)}, \tag{5}$$

by integrating both sides from initial state x_0, we get

$$\int_{x_0}^{x} \lambda(s) ds = \Lambda(x) = -\ln \frac{S(x)}{S(x_0)}, \tag{6}$$

leading to

$$S(x) = 1 - F(x) = S(x_0)e^{-\Lambda(x)}. \tag{7}$$

Based on the result of Lemma 2.3, we get:

$$x = \Lambda^{-1}(\ln \frac{S(x_0)}{1 - u}) \tag{8}$$

follows distribution $F(x)$. Because $1 - u$ also follows $U(0, 1]$, we get the result $x = \Lambda^{-1}(\ln \frac{S(x_0)}{u})$ follows distribution $F(x)$, and $S(x_0)e^{-\Lambda(x)}$ follows uniform distribution $U(0, 1]$. □

Right now, we lead to the proof of the main Theorem 2.1 and its main Corollary 2.1.1:

PROOF. For any agent x_i in the dynamical system $\mathscr{D}$, its arriving time t_i follows Poisson process $\mathscr{P}(t|\lambda_p) = \{t_1, ..., t_i, ...|0 < t_1 \le ... \le t_i \le .. \le t\}$ with $N(t|\lambda_t) = n$, then $\frac{t_i}{t}$ follows uniform distribution $U(0, 1]$ (Lemma 2.2). Replace $u = 1 - \frac{t_i}{t}$ in $F^{-1}(u)$ and $u = \frac{t_i}{t}$ in $\Lambda^{-1}(\ln \frac{S(x_0)}{u})$, get:

$$\begin{aligned} x_i(t) &= \int_{t_0}^{t} d\tau \frac{dF^{-1}(1 - \frac{t_i}{\tau})}{d\tau} = F^{-1}(1 - \frac{t_i}{t}) - F^{-1}(1 - \frac{t_i}{t_0}). \\ x_i(t) &= \int_{t_0}^{t} d\tau \frac{d\Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}\tau))}{d\tau} = \Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}t)). \end{aligned} \tag{9}$$

follows distribution $F(x(t))$, and also $F(x(t))$ has dynamicsïijŽ

$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{dF^{-1}(1 - \frac{t_i}{t})}{dt} = \frac{d\Lambda^{-1}(\ln(\frac{S(x_0)}{t_i}t))}{dt} \tag{10}$$

due to the equivalence described in Lemma 2.3 and 2.4. □

When $S(x_0) = 1$, above equation leads to:

$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{dF^{-1}(1 - \frac{t_i}{t})}{dt} = \frac{d\Lambda^{-1}(\ln(\frac{t}{t_i}))}{dt}. \tag{11}$$

2.4 Parameter Estimation

Due to the equivalence of the dynamics $\frac{dx_i(t)}{dt}$ and the distribution $f(x)$ described in the Theorem 2.1 and Corollary 2.1.1, we can learn their shared parameters by observations collected from either of them. Usually, we only get the cross-sectional observations $f(x)$. Thus, here we only show the parameter inference procedure with respect to cross-sectional data.

Given a set of cross-sectional data $\{x_1, ..., x_{n-1}, x_n\}_t$, we learn the parameters Θ of the distribution $f(x|\Theta)$, or $\lambda(x|\Theta)$ by maximizing their log-likelihood function:

$$\begin{aligned} \max_{\Theta} \ln L(\Theta|x_1, \ldots, x_n) &= \ln \prod_{i=1}^n f(x_i) \\ &= \ln \prod_{i=1}^n \lambda(x_i)e^{-\Lambda(x_i)} = \sum_{i=1}^n \ln \lambda(x_i) - \sum_{i=1}^n \Lambda(x_i) \end{aligned} \tag{12}$$

Table 1: ***Dynamical Origins of Distributions*****. By applying our Theorem 2.1 and Corollary 2.1.1, we showcase the discovered generative dynamics of 9 typical distributions in the upper table, and cross-sectional distributions generated by 9 typical dynamical systems in the lower table. The $f(x)$, $F(x)$, $\lambda(x)$ and $\Lambda(x)$ represents the probability density function, cumulative density function, hazard function and cumulative hazard function of random variable $x \geq x_0$ respectively. The *PA* is short for *P*referential *A*ttachment, *GC* for *G*rowth *C*ompetition, and *EL* for *E*nvironment *L*imit, which are possible interpretations for the dynamics in statistical physics and ecology. To the best of our knowledge, only the top four distributions, namely exponential, power law, stretched exponential (or Weibull) distributions and their generative dynamics are discussed [4].**

	Data Distribution		Survival Analysis		Dynamical system / Differential Equation		
	$f(x)$	$F(x)$	$\lambda(x)$	$\Lambda(x)$	$x_i(t)$	Dynamics $\frac{dx_i(t)}{dt}$	Interpretation
Exponential	$\alpha e^{-\alpha x}$	$1 - e^{-\alpha x}$	α	αx	$\frac{\ln(\frac{t}{t_i})}{\alpha}$	$\frac{1}{\alpha t}$	GC
Power law	$\alpha x_0^{\alpha} x^{-(\alpha+1)}$	$1 - (\frac{x_0}{x})^{\alpha}$	$\frac{\alpha}{x}$	$a \ln \frac{x}{x_0}$	$x_0(\frac{t}{t_i})^{\frac{1}{\alpha}}$	$\frac{x_i(t)}{\alpha t}$	PA + GC
Stretched exponential	$\frac{\alpha}{x^{\theta}} e^{-\frac{\alpha(x^{1-\theta}-x_0^{1-\theta})}{1-\theta}}$	$1 - e^{-\frac{\alpha(x^{1-\theta}-x_0^{1-\theta})}{1-\theta}}$	$\frac{\alpha}{x^{\theta}}$	$\frac{\alpha}{1-\theta}(x^{1-\theta} - x_0^{1-\theta})$	$[\ln(\frac{t}{t_i})\frac{1-\theta}{\alpha} + x_0^{1-\theta}]^{\frac{1}{1-\theta}}$	$\frac{x_i^{\theta}(t)}{\alpha t}$	Non-linear PA + GC
Weibull	$\alpha \lambda^{\alpha} x^{\alpha-1} e^{-(\lambda x)^{\alpha}}$	$1 - e^{-(\lambda x)^{\alpha}}$	$\alpha \lambda^{\alpha} x^{\alpha-1}$	$(\lambda x)^{\alpha}$	$\frac{(\ln \frac{t}{t_i})^{\frac{1}{\alpha}}}{\lambda}$	$\frac{x_i^{1-\alpha}(t)}{\lambda^{\alpha} \alpha t}$	Non-linear PA + GC
Log-logistic	$\frac{\lambda\alpha(\lambda x)^{\alpha-1}}{[1+(\lambda x)^{\alpha}]^2}$	$1 - \frac{1}{1+(\lambda x)^{\alpha}}$	$\frac{\lambda\alpha(\lambda x)^{\alpha-1}}{1+(\lambda x)^{\alpha}}$	$\ln[1 + (\lambda x)^{\alpha}]$	$\frac{(\frac{t}{t_i}-1)^{\frac{1}{\alpha}}}{\lambda}$	$\frac{x_i(t)}{\alpha(t-t_i)}$	PA + Since then GC
Sigmoid	$\frac{e^x}{(1+e^x)^2}$	$1 - \frac{1}{1+e^x}$	$\frac{e^x}{1+e^x}$	$\ln(1 + e^x)$	$\ln(\frac{t}{t_i} - 1)$	$\frac{1}{t-t_i}$	Since then GC
Log-normal *	$\frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}}$	$\Phi(\ln x)$	$\frac{f(x)}{1-\Phi(\ln x)}$	$-\ln[1 - \Phi(\ln x)]$	$e^{\Phi^{-1}(1-\frac{t_i}{t})}$	$x_i \frac{d\Phi^{-1}(z)}{dz} \frac{t_i}{t^2}$	PA + Square GC
Normal *	$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	$\Phi(x)$	$\frac{f(x)}{1-\Phi(x)}$	$-\ln[1 - \Phi(x)]$	$\Phi^{-1}(1 - \frac{t_i}{t})$	$\frac{d\Phi^{-1}(z)}{dz} \frac{t_i}{t^2}$	Square GC
Uniform	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{1}{b-x}$	$\ln \frac{b-a}{b-x}$	$b - (b-a)\frac{t_i}{t}$	$\frac{b-x_i(t)}{t}$	EL + GC
Generated by Exponential Dynamics	$\frac{\frac{\alpha}{t_i}}{x(\frac{\alpha}{t_i} \ln \frac{x}{x_0}+1)^2}$	$1 - \frac{1}{1+\frac{\alpha}{t_i} \ln \frac{x}{x_0}}$	$\frac{\frac{\alpha}{t_i}}{x(\frac{\alpha}{t_i} \ln \frac{x}{x_0}+1)}$	$\ln[\frac{\alpha}{t_i} \ln \frac{x}{x_0} + 1]$	$x_0 e^{\frac{t-t_i}{\alpha}}$	$\frac{x_i(t)}{\alpha}$	PA
Generated by Stretched Exponential Dynamics	$\frac{\frac{\alpha}{t_i^{1-\theta}}}{x[\frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{x}{x_0}+1]^{\frac{2-\theta}{1-\theta}}}$	$1 - [1 + \frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{x}{x_0}]^{\frac{-1}{1-\theta}}$	$\frac{\alpha}{\alpha(1-\theta)x \ln \frac{x}{x_0} + t_i^{1-\theta} x}$	$\frac{1}{1-\theta} \ln[\frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{x}{x_0} + 1]$	$x_0 e^{\frac{t^{1-\theta}-t_i^{1-\theta}}{\alpha(1-\theta)}}$	$\frac{x_i(t)}{\alpha t^{\theta}}$	PA+ Non-linear GC
Generated by Sigmoid Dynamics*	$\frac{d}{dx}(\frac{\frac{\alpha}{Nt_i} \ln A}{1+\frac{\alpha}{Nt_i} \ln A})$	$1 - \frac{1}{1+\frac{\alpha}{Nt_i} \ln A}$	$\frac{d}{dx}(\ln[\frac{\alpha}{Nt_i} \ln A + 1])$	$\ln[\frac{\alpha}{Nt_i} \ln A + 1]$	$N \frac{Be^{\frac{N(t-t_i)}{\alpha}}}{1+Be^{\frac{N(t-t_i)}{\alpha}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha}$	PA + EL
Generated by Log Logistic Dynamics*	$\frac{\alpha(\frac{N-x_0}{x_0})^{\frac{-\alpha}{N}} x^{-\frac{\alpha}{N}-1}}{N(N-x)^{\frac{-\alpha}{N}+1}}$	$1 - A^{-\frac{\alpha}{N}}$	$\frac{\frac{\alpha}{N}}{x(N-x)}$	$\frac{\alpha}{N} \ln A$	$N \frac{B(\frac{t}{t_i})^{\frac{N}{\alpha}}}{1+B(\frac{t}{t_i})^{\frac{N}{\alpha}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha t}$	PA + EL + GC
Generated by stretched Logistic Dynamics*	$-\frac{d}{dx}[1 + \frac{\alpha(1-\theta)}{Nt_i^{1-\theta}} \ln A]^{\frac{-1}{1-\theta}}$	$1 - [1 + \frac{\alpha(1-\theta)}{Nt_i^{1-\theta}} \ln A]^{\frac{-1}{1-\theta}}$	$\frac{d}{dx} \frac{\ln[1+\frac{\alpha(1-\theta)}{Nt_i^{1-\theta}} \ln A]}{1-\theta}$	$\frac{\ln[1+\frac{\alpha(1-\theta)}{Nt_i^{1-\theta}} \ln A]}{1-\theta}$	$\frac{NBe^{\frac{N}{\alpha}\frac{t^{1-\theta}-t_i^{1-\theta}}{1-\theta}}}{1+Be^{\frac{N}{\alpha}\frac{t^{1-\theta}-t_i^{1-\theta}}{1-\theta}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha t^{\theta}}$	PA + EL + Non-linear GC
Generated by Confined Exponential	$\frac{\frac{\alpha}{t_i}\frac{1}{N-x}}{(1-\frac{\alpha}{t_i} \ln \frac{N-x}{N-x_0})^2}$	$1 - \frac{1}{1-\frac{\alpha}{t_i} \ln \frac{N-x}{N-x_0}}$	$\frac{\frac{\alpha}{t_i}\frac{1}{N-x}}{1-\frac{\alpha}{t_i} \ln \frac{N-x}{N-x_0}}$	$\ln[1 - \frac{\alpha}{t_i} \ln \frac{N-x}{N-x_0}]$	$N - \frac{N-x_0}{e^{\frac{(t-t_i)}{\alpha}}}$	$\frac{N-x_i(t)}{\alpha}$	EL
Generated by Confined Power law	$\frac{\alpha(N-x_0)^{-\alpha}}{(N-x)^{1-\alpha}}$	$1 - (\frac{N-x}{N-x_0})^{\alpha}$	$\frac{\alpha}{N-x}$	$-\alpha \ln \frac{N-x}{N-x_0}$	$N - (N - x_0)(\frac{t}{t_i})^{\frac{-1}{\alpha}}$	$\frac{N-x_i(t)}{\alpha t}$	GC + EL
Generated by Confined Stretched Exponentail	$\frac{\frac{\alpha}{t_i^{1-\theta}(N-x)}}{[1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{N-x}{N-x_0}]^{\frac{2-\theta}{1-\theta}}}$	$1 - [1 - \frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{N-x}{N-x_0}]^{\frac{-1}{1-\theta}}$	$\frac{\frac{\alpha}{t_i^{1-\theta}(N-x)}}{1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{N-x}{N-x_0}}$	$\frac{\ln[1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}} \ln \frac{N-x}{N-x_0}]}{1-\theta}$	$N - \frac{(N-x_0)}{e^{\frac{t^{1-\theta}-t_i^{1-\theta}}{\alpha(1-\theta)}}}$	$\frac{N-x_i(t)}{\alpha t^{\theta}}$	Non-linear GC + EL
Generated by Linear Dynamics	$\frac{\frac{\alpha}{t_i}}{[\frac{\alpha}{t_i}(x-x_0)+1]^2}$	$1 - \frac{1}{\frac{\alpha}{t_i}(x-x_0)+1}$	$\frac{1}{(x-x_0)+\frac{t_i}{\alpha}}$	$\ln[\frac{\alpha}{t_i}(x - x_0) + 1]$	$x_0 + \frac{t-t_i}{\alpha}$	$\frac{1}{\alpha}$	Constant rate

Notes: t_i in the distribution means that the generated distribution is time-varing as dynamical system evolves. * $z = 1 - \frac{t_i}{t}$; $A = \frac{N-x_0}{x_0}\frac{x}{N-x}$, and $B = \frac{x_0}{N-x_0}$.

according to the parametric forms of $f(x|\Theta)$ and the relationship $f(x|\Theta) = \lambda(x|\Theta)e^{-\Lambda(x|\Theta)}$. Then, the estimated generative dynamics is $\frac{d\hat{x}_i(t|\hat{\Theta})}{dt}|_{x_0} = \frac{d\Lambda^{-1}(\ln(\frac{S(x_0|\hat{\Theta})}{t_i}t)|\hat{\Theta})}{dt}$. We show that sometimes it's more handy to use the Corollary 2.1.1 with respect to the survival function and the hazard function in Section 4 *Learning Dynamics from Empirical Data*.

2.5 Generator

Data samples can be generated through either dynamical system side or distribution side. To generate cross-sectional data samples of $x_i(t)$ $i = 1, 2, \ldots$ given generative dynamics $\frac{dx_i(t)}{dt}|_{x_0}$ is straightforward by applying Theorem 2.1, namely observing $x_i(t) = \int_{t_0}^{t} \frac{dx_i(\tau)}{d\tau} d\tau$ at time t where arriving time t_i of each agent i follows Poisson process $\mathscr{P}(t|\lambda_p) = \{t_1, \ldots, t_i, \ldots | 0 < t_1 \leq \ldots \leq t_i \leq \ldots \leq t\}$ and $i = 1, \ldots, N(t|\lambda_t)$.

On the other hand, to generate data samples from distribution side, especially the cumulative hazard rate, the method is based on solving the equation $\ln(u) + \Lambda(x) = \ln S(x_0)$ for x, where u is generated from uniform distribution $U(0, 1]$. We can apply the most general method, like Newton's iterative method to solve above equation. When we have $F(x)$, $S(x)$ or $\Lambda(x)$, we can get samples by using Lemma 2.3 or Lemma 2.4.

3 DISCOVERING DYNAMICS AND DISTRIBUTIONS

By applying our Theorem 2.1 and Corollary 2.1.1, we discover *(i)* generative dynamics of given distributions, and *(ii)* new (cross-sectional) distributions generated by given dynamical systems at time t. *(iii)* Interpretable patterns of dynamical systems, namely "mechanisms", are summarized. *(iv)* New distributions with needed dynamic properties can be designed in a principled way.

In a word, we try to find intrinsic relationships between the cross-sectional distributions, hazard functions in survival analysis and their longitudinal generative dynamics. We showcase our results in Table 1. Key steps to derive the table are shown in Table 3.

3.1 Discovering Longitudinal Generative Dynamics

We have discovered some new generative dynamics of typical distributions by the Theorem 2.1 and Corollary 2.1.1 (Refer to Table 1). To the best of our knowledge, only exponential, power law, stretched exponential (or Weibull) distributions and their generative dynamics are discussed case by case [4].

- **Power-law and Exponential distributions.** Power-law distribution $f(x) = \alpha x_0^{\alpha} x^{-(\alpha+1)}$ where $x \geq x_0$ is famous for its scaling property and generative mechanisms [5]. Its hazard rate is in a much simpler form $\frac{\alpha}{x}$. By using the Corollary 2.1.1 , we get the generative growth curve $x_i(t) = x_0(\frac{t}{t_i})^{\frac{1}{\alpha}}$ and the dynamic differential equation $\frac{dx_i(t)}{dt} = \frac{x_i(t)}{\alpha t}$ (shown in Table 1). As for the studies on complex network (statistical physics), the $x_i(t)$ represents the degree of node i, and $\frac{dx_i(t)}{dt} \propto x_i(t)$ means the preferential attachment mechanism of network evolution. The αt represents the growth competition due to the arriving of new nodes. In the scenario of economy, the $x_i(t)$ represents the wealth and the $\frac{dx_i(t)}{dt} \propto x_i(t)$ means the rich-get-richer (or Matthew) phenomena. The αt represents the growth of the competition due to the new competitors pouring into the market. Similar dynamic ideas are applied to Dirichlet process. In contrast, the exponential distribution $f(x) = \alpha e^{-\alpha x}$ has generative dynamics $\frac{dx_i(t)}{dt} = \frac{1}{\alpha t}$, without the preferential attachment term $x_i(t)$ and the growth rate decaying linearly due to the growth competition, which corresponds to the random graph scenario.
- **Stretched-exponential or Weibull distributions.** For the stretched exponential $f(x) = \frac{\alpha}{x^{\theta}} e^{-\frac{\alpha}{1-\theta}(x^{1-\theta})}$ and Weibull $f(x) = \alpha\lambda^{\alpha} x^{\alpha-1} e^{-(\lambda x)^{\alpha}}$, they share the same dynamics, namely $\frac{dx_i(t)}{dt} = \frac{x_i^{\theta}(t)}{\alpha t}$ and $\frac{dx_i(t)}{dt} = \frac{x_i^{1-\alpha}(t)}{\lambda^{\alpha}\alpha t}$. Compared with the power law distribution, they have non-linear preferential attachment growth dynamics $x_i^{\theta}(t)$, tuned by a power exponent θ.
- **Sigmoid and Log-logistic distributions.** Sigmoid distribution $f(x) = \frac{e^x}{(1+e^x)^2}$ is widely used in deep learning as an activation function. Our theorem explains its generative dynamics as $\frac{dx_i(t)}{dt} = \frac{1}{t-t_i}$, featuring a singular burst (of rate) at the birth time t_i and then growing $\ln(\frac{t}{t_i} - 1)$ similar to the exponential case $\ln(\frac{t}{t_i})/\alpha$. As for the log-logistic distribution $f(x) = \frac{\lambda\alpha(\lambda x)^{\alpha-1}}{[1+(\lambda x)^{\alpha}]^2}$, its generative dynamics $\frac{dx_i(t)}{dt} = \frac{x_i(t)}{\alpha(t-t_i)}$, which also has the singular burst rate at birth time t_i but with a much faster growth due to the linear preferential-attachment term $x_i(t)$.
- **Log-normal and Normal distributions.** Log-normal distribution $f(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}}$ and normal distribution $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ have the inferred generative dynamics $\frac{dx_i(t)}{dt} = x_i(t)\frac{d\Phi^{-1}(z)}{dz}\frac{t_i}{t^2}$ and $\frac{dx_i(t)}{dt} = \frac{d\Phi^{-1}(z)}{dz}\frac{t_i}{t^2}$ respectively, where $z = 1 - \frac{t_i}{t}$. They both decay with squared growth competition, but the log-normal has the linear preferential attachment term $x_i(t)$, implying the heavy-tailed property.
- **Uniform distribution.** The uniform distribution $f(x) = \frac{1}{b-a}$ corresponds to the generative dynamic $x_i(t) = b - (b-a)\frac{t_i}{t}$, which follows constrained exponential growth as discussed later. The differential rate $\frac{dx_i(t)}{dt} = \frac{b - x_i(t)}{t}$ features an environment limit term $b - x_i(t)$ and growth competition term $\frac{1}{t}$.

3.2 Discovering Cross-sectional Distributions

Next, we get some new cross-sectional distributions generated from dynamical systems with interpretable mechanisms [2] (Refer to Table 1):

- **Exponential growth.** Due to the fast growth rate $\frac{dx_i(t)}{dt} = \frac{x_i(t)}{\alpha}$, the uniform inputs lead to very heterogeneous outputs transformed by an exponential growth system, exhibiting a log-power-law distribution (or log-Cauchy distribution)

$f(x(t_i)) = \frac{\frac{\alpha}{t_i}}{x(\frac{\alpha}{t_i}\ln\frac{x}{x_0}+1)^2}$. The t_i in $f(x)$ implies that the $f(x)$ is changing (unstable) over time. The exponential growth can be explained by a branching process where $\frac{1}{\alpha}$ is the average branching factor, the growth of a full tree, and the division of a cell at an early stage.

- **Power-Law and Stretched-Exponential growth.** The power-law growth dynamics $\frac{dx_i(t)}{dt} = \frac{x_i(t)}{\alpha t}$ and the stretched-exponential growth dynamics $\frac{dx_i(t)}{dt} = \frac{x_i(t)}{\alpha t^\theta}$ [23], featuring preferential attachment $x_i(t)$ and linear- or non-linear-growth competitions respectively. The power-law distribution $f(x) = \alpha x_0^\alpha x^{-(\alpha+1)}$ is generated by power-law growth dynamics, while the log-power-law distribution $f(x) = \frac{\frac{\alpha}{t_i^{1-\theta}}}{x[\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{x}{x_0}+1]^{\frac{2-\theta}{1-\theta}}}$ with tuned exponent θ is generated by the stretched-exponential dynamics.
- **Sigmoid, Log-Logistic, and Stretched Log-Logistic growth.** Following the logistic growth framework (preferential attachment term times environment limit term $x_i(t) * [N - x_i(t)]$) but with different rate of the growth competition, the Sigmoid growth $\frac{dx_i(t)}{dt} = \frac{x_i(t)[N-x_i(t)]}{\alpha}$, log-logistic growth $\frac{dx_i(t)}{dt} = \frac{x_i(t)[N-x_i(t)]}{\alpha t}$, and the stretched-log-logistic growth $\frac{dx_i(t)}{dt} = \frac{x_i(t)[N-x_i(t)]}{\alpha t^\theta}$, generate complex logistic form distributions as shown in Table 1. Taking the log-logistic growth $\frac{dx_i(t)}{dt} = \frac{x_i(t)[N-x_i(t)]}{\alpha t}$ dynamics as an example, we can interpret it as a growing scale-free network model with degree constraints on the hub nodes, leading to more hubs with moderate degree rather than few oligarchs.
- **Confined Exponential, Confined Power-Law, and Confined Stretched-Exponential growth.** Following the confined growth framework (environment limit term $N - x_i(t)$) but with different rate of the growth competition, the confined-exponential growth $\frac{dx_i(t)}{dt} = \frac{N-x_i(t)}{\alpha}$, confined-power-law growth $\frac{dx_i(t)}{dt} = \frac{N-x_i(t)}{\alpha t}$, and the confined stretched-exponential growth $\frac{dx_i(t)}{dt} = \frac{N-x_i(t)}{\alpha t^\theta}$, generate complex confined-form distributions as shown in Table 1. The uniform distribution is generated from a special case of the confined power law growth.

3.3 Interpretations and Implications

We further summarize common generative patterns of the cross-sectional distributions by our theorem. For example, faster the growth dynamics is, more heavy-tailed the distribution is, implying a large data variance. We find the preferential attachment term $\frac{dx_i(t)}{dt} \propto x_i^\theta(t)$ is the common ingredient for the **heavy-tailed** distribution, like power-law, stretched-exponential (or Weibull), log-normal, log-logistic, log-power-law etc. as shown in Table 1. In contrast, the faster the growth competition $\frac{dx_i(t)}{dt} \propto \frac{1}{t^\theta}$ is, like the square competition in the dynamics $\frac{dx_i(t)}{dt} = \frac{d\Phi^{-1}(z)}{dz}\frac{t_i}{t^2}$ of normal distribution and the linear competition in the dynamics $\frac{dx_i(t)}{dt} = \frac{1}{\alpha t}$ of exponential distribution, the narrower the tail is, and thus less variance.

On the other hand, distributions can be designed by their generative dynamics. For example, the since-then-growth-competition $\frac{1}{t-t_i}$ can be used to design activation function with an initial burst, the environment limit $N - x_i(t)$ and growth competition mechanisms can be used to add constraints on the variance. Other ingredients like the preferential attachment is used to generate heavy-tailed distributions, and non-linear power exponent is used to add model flexibility, and the combinations of above mechanisms can be utilized to construct totally new and complex distributions with needed dynamical properties.

4 LEARNING DYNAMICS FROM EMPIRICAL DATA

In the previous section, we show how to infer the generative dynamics from distributions and how to get cross-sectional distributions from generative dynamics by our theorem. In this section, we showcase a simple but versatile model by applying our theorem to learn the dynamics directly from empirical data samples, together with distribution fitting and sample generation procedures. More complex models can be formulated in a similar way.

Applying our Theorem 2.1 and Corollary 2.1.1, we showcase a parametric model to capture complex empirical distributions, which is specified by the hazard function:

$$\lambda(x) = \beta + \frac{\alpha}{(x+\Delta)^\theta} \tag{13}$$

where $X > -\Delta$. The corresponding probability density function can be derived by the equation $f(x) = \lambda(x)e^{-\int_{x_0}^{x}\lambda(s)ds}$. We get the probability density function when $\theta \neq 1$:

$$\begin{aligned} f(x) &= \beta e^{-\beta x - \frac{\alpha}{1-\theta}[(x+\Delta)^{1-\theta} - \Delta^{1-\theta}]} \\ &+ \alpha(x+\Delta)^{-\theta} e^{-\beta x - \frac{\alpha}{1-\theta}[(x+\Delta)^{1-\theta} - \Delta^{1-\theta}]} \end{aligned} \tag{14}$$

and the probability density function when $\theta = 1$ is:

$$f(x) = \beta e^{-\beta x}(\frac{x}{\Delta}+1)^{-\alpha} + \frac{\alpha}{\Delta}(\frac{x}{\Delta}+1)^{-(\alpha+1)}e^{-\beta x} \tag{15}$$

Equation 13 is a *heavy-tailed mixture model* which encompasses a wide range of distributions [1], including narrow-tailed, heavy-tailed, and complex mixture distributions. Their generative dynamics is derived by applying Theorem 2.1 (where $x_0 = 0$ and $S(x_0) = 1$):

$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{(x_i(t)+\Delta)^\theta}{\beta(x_i(t)+\Delta)^\theta t + \alpha t}, \tag{16}$$

serving as the dynamic origin of a complex heavy-tailed-mixture model we formulate:

- When $\theta \neq 1$ and $\beta = 0$, the $f(x)$ degenerates to a stretched exponential (or Weibull) distribution $f(x|\beta = 0) = \alpha(x+\Delta)^{-\theta}e^{\frac{\alpha}{1-\theta}[(x+\Delta)^{1-\theta}-\Delta^{1-\theta}]}$ [2], we get special-case generative dynamics:
$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{(x_i(t)+\Delta)^\theta}{\alpha t} \tag{17}$$
- When $\theta = 1$ and $\beta = 0$, we get the generative dynamics of distributions which show power-law tail, namely $f(x) =$

[1] Without loss of generality, we here showcase a simple but versatile model. More complex parametric models and their generative dynamics, e.g., $\lambda(x) = \sum_i \frac{\alpha_i}{(x+\Delta_i)^{\theta_i}}$, can be analyzed by our Theorem in a principled way.
[2] Encompassing Gaussian-like distributions as special cases when $\theta = -1$.

$\frac{\alpha}{\Delta}(\frac{x}{\Delta}+1)^{-(\alpha+1)}$ as a special case:

$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{x_i(t)+\Delta}{\alpha t} \tag{18}$$

- When $\alpha = 0$, we get the generative dynamics of the exponential distribution $f(x|\alpha=0) = \beta e^{-\beta x}$ as a special case:

$$\frac{dx_i(t)}{dt}|_{x_0} = \frac{1}{\beta t} \tag{19}$$

We learn the modeling parameters by the maximum likelihood estimation framework. When given the hazard function Equ. 13, we get the log-likelihood function as:

$$\begin{aligned}\ln L(x_1,\dots,x_n) = &\sum_{i=1}^{n} \ln\left[\beta + \alpha(x_i+\Delta)^{-\theta}\right] - \beta\sum_{i=1}^{n} x_i \\ &- \frac{\alpha}{1-\theta}\sum_{i=1}^{n}\left[(x_i+\Delta)^{1-\theta} - \Delta^{1-\theta}\right]\end{aligned} \tag{20}$$

Thus, general optimization algorithms with/without gradients like the the interior point algorithm [6] or the Levenberg-Marquardt algorithm [13] can be used for maximizing objective function 20 . Specific optimization algorithms can be further designed based on the choice of the distribution or dynamics.

Sample generators based on above hazard function and dynamic differential equations are given as follows:

Input : Hazard function of $\lambda(x) = \beta + \frac{\alpha}{x+\Delta}$, total event number n
Output: $\{x_1, \dots, x_n\}$
Set current iteration $i = 1$;
while $i \le n$ **do**
  Sample $u \sim U(0,1]$;
  Solve $\log u + \int_{x_0}^{x} \lambda(s)ds = 0$ for x by Algorithm 3.;
  $x_i = x$;
  i += 1 ;
end

Algorithm 1: Generating samples from hazard function Eq. 13

Input : Dynamic equation Eq. 13, Poisson process $\mathscr{P}(t|\lambda_p) = \{t_1, \dots, t_i, \dots | t_1 \le \dots \le t_i \le .. \le t\}$ with $N(t|\lambda_t) = n$,
Output: $\{x_1(t), \dots, x_n(t)\}$
Set current iteration $i = 1$;
while $i \le n$ **do**
  if $\theta == 1$ **then**
    Solve $\beta x_i(t) + \alpha\ln(\frac{x_i(t)}{\Delta}+1) = \ln\frac{t}{t_i}$ for $x_i(t)$ by Algorithm 3.;
  else
    Solve $\beta x_i(t) + \frac{\alpha[(x_i(t)+\Delta)^{1-\theta} - \Delta^{1-\theta}]}{1-\theta} = \ln\frac{t}{t_i}$ for $x_i(t)$ by Algorithm 3. (Equation solver in *Supplemental Material Sec.D*);
  end
  i += 1 ;
end

Algorithm 2: Generating samples from dynamic differential Eq. 16

5 EXPERIMENTS

We validate our Theorem 2.1 and the Corollary 2.1.1 together with parameter inference and simulation algorithms on both synthetic and real-world datasets by answering the following questions:

- Can the dynamical systems generate cross-sectional distributions predicted by our theorem?
- Can we learn the parameters of the generated distributions?

Input : Equation $\Phi(x) = \log u + \int_{x_0}^{x} \lambda(s)ds$.
Output: x
Set $\epsilon = 10^{-8}$, $x = 0$;
while $|\Phi(x)| \le \epsilon$ **do**
  if $\theta == 1$ **then**
    $\Phi(x) = \ln u + \beta x + \alpha\ln(\frac{x}{\Delta}+1)$;
  else
    $\Phi(x) = \ln u + \beta x + \frac{\alpha}{1-\theta}[(x+\Delta)^{1-\theta} - \Delta^{1-\theta}]$;
  end
  $\Phi'(x) = \beta + \alpha(x+\Delta)^{-\theta}$;
  $x = x - \frac{\Phi(x)}{\Phi'(x)}$;
end

Algorithm 3: Equation solver: Newton's iterative method

- Can we recover the generative dynamics from its cross-sectional distributions?
- What are the plausible generative dynamics of a wide range of real-world datasets?

Moreover, to the best of our knowledge, there is no such a theorem serving as our baselines which describies the relationships between a dynamical system and the distribution of its cross-sectional states.

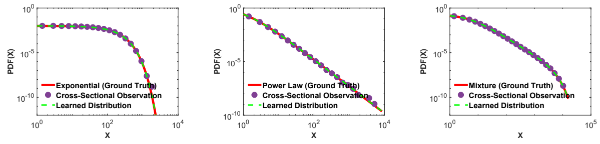
5.1 Synthetic data

Table 2: Experiment Configurations. Generative dynamics and their cross-sectionally generated distributions.

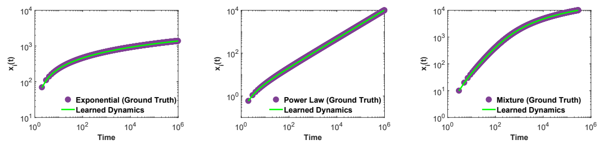
	Dynamics $\frac{dx_i(t)}{dt}\|_{x_0}$	PDF $f(x)$	Parameters
Exponential	$\frac{1}{\beta t}$	$\beta e^{\beta x}$	$\beta = 0.01$
Power Law	$\frac{x_i(t)+\Delta}{\alpha t}$	$\alpha\Delta^{\alpha}x^{-(\alpha+1)}$	$\alpha = 1.5$, $\Delta = 1$
Mix model	$\frac{x_i(t)+\Delta}{\beta(x_i(t)+\Delta)t+\alpha t}$	$\beta e^{-\beta x}(\frac{x}{\Delta}+1)^{-\alpha} + \frac{\alpha}{\Delta}(\frac{x}{\Delta}+1)^{-(\alpha+1)}e^{-\beta x}$	$\beta = 5e{-}4$, $\alpha = 1$, $\Delta = 5$

Experiment setup. Table 2 lists the ground-truth setting with three differential equations to capture three different underlying generative dynamics, whose cross-sectional distribution follows exponential distribution (narrow-tailed), power-law distribution (heavy-tailed), and heavy-tailed mixture distribution respectively predicted by the Theorem 2.1. For each dynamical system, we set $E[N(t|\lambda_t)] = 10^6$ agents in the time interval $(0, 10^6]$, namely $\mathscr{P}(t|\lambda_p) = \{t_1, \dots, t_i, \dots | 0 < t_1 \le \dots \le t_i \le .. \le t\}$ where $t = 10^6$ and $\lambda_t = 1$. The 10^6 agents change their states according to dynamics $\frac{dx_i(t)}{dt}|_{x_0=0}$ as shown in Table 2 and we observe their cross-sectional states at time $t = 10^6$.

Results. The dynamics indeed generate cross-sectional observations with predicted distributions by the Theorem 2.1. As shown in Fig. 2 a-c, the distribution of generated states (purple dots) fits the ground truth distribution (red lines) exactly for all the model configurations, ranging from narrow-tailed distribution (Fig. 2(a)), fat-tailed distribution(Fig. 2(b)), to mixture of fat-tailed distributions (Fig. 2(c)). We further test the null hypothesis that the observational data and the ground truth distribution are from the same continuous distribution by using the two-sample Kolmogorov-Smirnov test [20] at the 1% significance level. We accept the null hypothesis at the the 1% significance level for all the settings, with p-value



(a) Exp. Distribution (b) PL. Distribution (c) Mix. Distribution



(d) Exp. Generative Dynamics (e) PL. Generative Dynamics (f) Mix. Generative Dynamics

Figure 2: All the dynamical systems generate cross-sectional samples with predicted distributions. Our model accurately recovers the parameters of these dynamics, and the generators reproduce realistic data samples. Exp: Exponential, PL:Power-Law, Mix: Mixture. All the figures are on log-log plots.

(KS-distance) 0.99 ($4.7 * 10^{-4}$), 0.51 ($1.2 * 10^{-3}$), and 0.78 ($6.9 * 10^{-4}$) for three settings respectively.

Our inference framework can learn the modeling parameters from the cross-sectional data samples. By maximizing the log-likelihood function indicated by Eq. 12 and Eq. 20, we get $\hat{\beta} = 9.988 * 10^{-3}$ for exponential case, $\hat{\alpha} = 1.499, \hat{\Delta} = 0.999$ for the power-law case, and $\hat{\beta} = 4.969 * 10^{-4}, \hat{\alpha} = 1.000, \hat{\Delta} = 5.005$ for the mixture model case, reaching the true value accurately as shown in Table 2.

We indeed recover the generative dynamics. By learning the parameters from observed distributions, we infer the generative dynamics and then generate the estimated growth dynamics $\hat{x_1}(t)$. As shown in Fig. 2 d-f, the estimated dynamics (green lines) fit the true dynamics (purple dots) accurately.

5.2 Empirical data

We examine a wide range of real-world datasets from different disciplines to answer what are their plausible generative dynamics. In reality, the paradox is that sometimes we never know the real generative dynamics. Thus, we evaluate our model by checking if our model can fit and reproduce the reality we observed [27]. The datasets are: *(a)* The number of occurrence of words in the novel Moby Dick by Herman Melville [15]; *(b)* The number of deaths by terrorist attacks worldwide from February 1968 to June 2006 [7]; *(c)* The inter-event-time of adding consecutive friends in WeChat (largest social network in China) by an active user [24, 25]; *(d)* The time interval of two retweets in a information cascade in Tencent Weibo [26];*(e)* The time interval of chatting behavior in an online group from Tencent QQ [29]; and *(f)* The response time of letter correspondence for Einstein during his whole life [16].

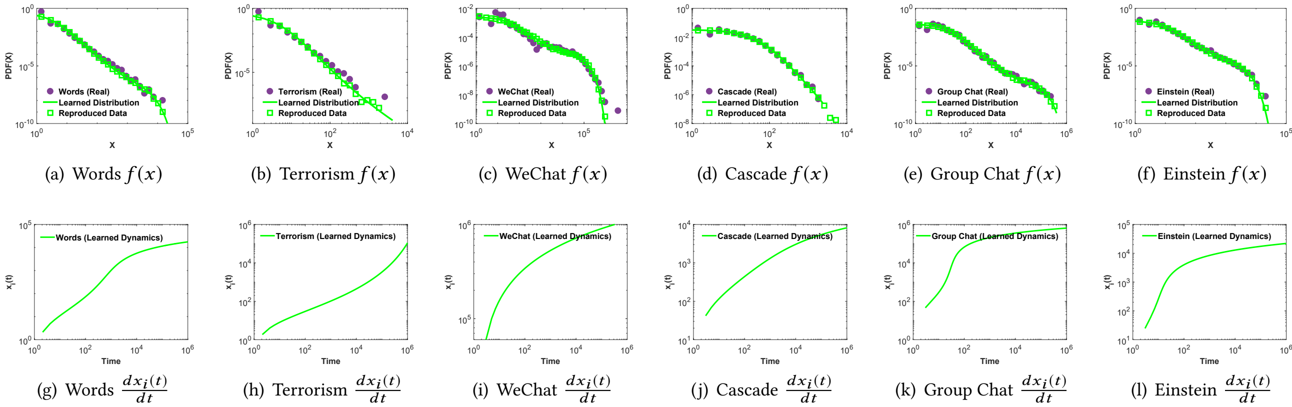
Actually, distributions in the real world exhibit large complexity as shown in Fig. 3. For example, the correspondence dynamics of Einstein in Fig. 3f , we find: flat part when x (response time) is small, implying Poisson (-like) process at short-time scale; heavy-tailed x in the middle-time scale, implying a (priority-queue based) decision-making process; and a bimodal part in the long-time scale, implying a routinely Poisson (-like) process at long-time scale. Besides, we find the inferred generative dynamics for real-world datasets are highly non-linear and complex. For the Einstein case Fig. 3l, accelerating dynamics and then followed by a saturated part, implying the intrinsic complex dynamics of Einstein's lifetime correspondence dynamics. In contrast, the dynamics of the global terrorist attacks possibly follow consistently accelerating growth as shown in Fig. 3h.

Though the intrinsic complexities in the datasets, we capture them accurately by our framework and we plot the estimated distribution in Fig. 3. In Fig. 3a-f, the estimated distributions (green lines) capture the complex real distributions (purple dots) in all these datasets. Further, we show their inferred dynamics in Fig. 3g-l. We reproduce the data samples by observing these estimated dynamics at time t and plot the distributions of the generated data in Fig. 3a-f shown in green squares. We find the generated data well capture the reality in all these widely different datasets. Specifically, we get very small KS-distance for all the datasets: (a) $3.19 * 10^{-1}$, (b) $3.48*10^{-1}$, (c) $7.60*10^{-2}$, (d) $2.10*10^{-2}$, (e) $8.20*10^{-2}$, (f) $7.60*10^{-2}$, implying that we learned their plausible generative dynamics.

6 CONCLUSION

Many laws of nature are of dynamic origins, however, what we can observe are always cross-sectional states at a specific time point. Thus, a lot of scientific efforts/literature follow this research approach: *to learn the evolutionary dynamics of a complex system from its cross-sectional states at a specific time point.* Here we try to build the first theoretical relationships between them. We treat distribution function of datasets as cross-sectional states generated by a dynamical system in our Theorem 2.1 and Corollary 2.1.1, which build the equivalence between a cross-sectional distribution (or in the form of hazard function) and its generative dynamics theoretically. We then propose parameter estimation and simulation algorithms. By applying our theory, we have discovered many new generative dynamics of distributions, and new distributions generated by given dynamical systems. We further showcase a model to learn the generative dynamics directly from empirical data samples. We accurately capture various synthetic and complex real-world datasets. Our study may potentially facilitate scientific discoveries of the unknown generative dynamics of complex systems which generate complex cross-sectional data in the real world.

Limitation and extensions. We can extend our Theorem 2.1 and Corollary 2.1.1 to the non-absolutely continuous cases: the cumulative hazard function can be generalized as $\Lambda(x) = -\int_{x_0}^{x} \frac{dS(t)}{S(t-)}$ to non-absolutely continuous case due to the fact that S(x) is monotonously decreasing [1]. The extension to the non-parametric cases can be based on the non-parametric estimates for the $\Lambda(x)$ (e.g. Nelson-Aalen estimator) and $S(x)$ (e.g. Kaplan-Meier estimator) [1]. Semi-parametric models which incorporate covariates remains to be explored.



(a) Words $f(x)$ (b) Terrorism $f(x)$ (c) WeChat $f(x)$ (d) Cascade $f(x)$ (e) Group Chat $f(x)$ (f) Einstein $f(x)$

(g) Words $\frac{dx_i(t)}{dt}$ (h) Terrorism $\frac{dx_i(t)}{dt}$ (i) WeChat $\frac{dx_i(t)}{dt}$ (j) Cascade $\frac{dx_i(t)}{dt}$ (k) Group Chat $\frac{dx_i(t)}{dt}$ (l) Einstein $\frac{dx_i(t)}{dt}$

Figure 3: ***The empirical PDFs, their distribution fitting results, learned generative dynamics and reproduced samples by observing the dynamics cross-sectionally at time*** 10^6 ***for*** 6 ***real-world datasets from a wide range of scenarios. The empirical distributions a-f and their inferred generative dynamics g-l are complex, but our method fits them accurately. All the figures are on log-log plots.***

Acknowledgments

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Table 3: ***Appendix: Dynamical Origins of Distributions - Key Derivation Steps.*** **By applying our Theorem 2.1 and Corollary 2.1.1, we showcase the discovered generative dynamics of 9 typical distributions in the upper table, and the discovered distributions generated by 9 typical dynamical systems in the lower table. The $f(x)$, $\lambda(x)$ and $\Lambda(x)$ represent the probability density function, hazard function and cumulative hazard function of random variable $x \geq x_0$ respectively.** ***We add Λ^{-1} column here to illustrate the key steps from $f(x)$ and $\lambda(x)$ to $\frac{dx}{dt}$ by applying the Corollary 2.1.1.*** **The *PA* is short for *P*referential *A*ttachment, *GC* for *G*rowth *C*ompetition, and *EL* for *E*nvironment *L*imit, which are possible interpretations for the dynamics in statistical physics and ecology.**

	Data Distribution	Survival Analysis			Dynamical System		
	$f(x)$	$\lambda(x)$	$\Lambda(x)$	$\Lambda^{-1}(y)$	$x_i(t)$	Dynamics $\frac{dx_i(t)}{dt}$	Interpretation
Exponential	$\alpha e^{-\alpha x}$	α	αx	$\frac{y}{\alpha}$	$\frac{\ln(\frac{t}{t_i})}{\alpha}$	$\frac{1}{\alpha t}$	GC
Power law	$\alpha x_0^{\alpha} x^{-(\alpha+1)}$	$\frac{\alpha}{x}$	$a \ln \frac{x}{x_0}$	$x_0 e^{\frac{y}{\alpha}}$	$x_0(\frac{t}{t_i})^{\frac{1}{\alpha}}$	$\frac{x_i(t)}{\alpha t}$	PA + GC
Stretched exponential	$\frac{\alpha}{x^{\theta}} e^{-\frac{\alpha}{1-\theta}(x^{1-\theta}-x_0^{1-\theta})}$	$\frac{\alpha}{x^{\theta}}$	$\frac{\alpha}{1-\theta}(x^{1-\theta} - x_0^{1-\theta})$	$[(\frac{1-\theta}{\alpha})y + x_0^{1-\theta}]^{\frac{1}{1-\theta}}$	$[\ln(\frac{t}{t_i})\frac{1-\theta}{\alpha} + x_0^{1-\theta}]^{\frac{1}{1-\theta}}$	$\frac{x_i^{\theta}(t)}{\alpha t}$	Non-linear PA + GC
Weibull	$\alpha\lambda^{\alpha} x^{\alpha-1} e^{-(\lambda x)^{\alpha}}$	$\alpha\lambda^{\alpha} x^{\alpha-1}$	$(\lambda x)^{\alpha}$	$\frac{y^{\frac{1}{\alpha}}}{\lambda}$	$\frac{(\ln\frac{t}{t_i})^{\frac{1}{\alpha}}}{\lambda}$	$\frac{x_i^{1-\alpha}(t)}{\lambda^{\alpha}\alpha t}$	Non-linear PA + GC
Log-logistic	$\frac{\lambda\alpha(\lambda x)^{\alpha-1}}{[1+(\lambda x)^{\alpha}]^2}$	$\frac{\lambda\alpha(\lambda x)^{\alpha-1}}{1+(\lambda x)^{\alpha}}$	$\ln[1 + (\lambda x)^{\alpha}]$	$\frac{(e^y-1)^{\frac{1}{\alpha}}}{\lambda}$	$\frac{(\frac{t}{t_i}-1)^{\frac{1}{\alpha}}}{\lambda}$	$\frac{x_i(t)}{\alpha(t-t_i)}$	PA + Since then GC
Sigmoid	$\frac{e^x}{(1+e^x)^2}$	$\frac{e^x}{1+e^x}$	$\ln(1 + e^x)$	$\ln(e^y - 1)$	$\ln(\frac{t}{t_i} - 1)$	$\frac{1}{t-t_i}$	Since then GC
Log-normal *	$\frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}}$	$\frac{f(x)}{1-\Phi(\ln x)}$	$-\ln[1 - \Phi(\ln x)]$	$e^{\Phi^{-1}(1-e^{-y})}$	$e^{\Phi^{-1}(1-\frac{t_i}{t})}$	$x_i \frac{d\Phi^{-1}(z)}{dz}\frac{t_i}{t^2}$	PA + Square GC
Normal *	$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	$\frac{f(x)}{1-\Phi(x)}$	$-\ln[1 - \Phi(x)]$	$\Phi^{-1}(1 - e^{-y})$	$\Phi^{-1}(1 - \frac{t_i}{t})$	$\frac{d\Phi^{-1}(z)}{dz}\frac{t_i}{t^2}$	Square GC
Uniform	$\frac{1}{b-a}$	$\frac{1}{b-x}$	$\ln\frac{b-a}{b-x}$	$b - (b-a)e^{-y}$	$b - (b-a)\frac{t_i}{t}$	$\frac{b-x_i(t)}{t}$	EL + GC
Generated by Exponential Dynamics	$\frac{\frac{\alpha}{t_i}}{x(\frac{\alpha}{t_i}\ln\frac{x}{x_0}+1)^2}$	$\frac{\frac{\alpha}{t_i}}{x(\frac{\alpha}{t_i}\ln\frac{x}{x_0}+1)}$	$\ln[\frac{\alpha}{t_i}\ln\frac{x}{x_0} + 1]$	$x_0 e^{\frac{t_i}{\alpha}(e^y-1)}$	$x_0 e^{\frac{t-t_i}{\alpha}}$	$\frac{x_i(t)}{\alpha}$	PA
Generated by Stretched Exponential Dynamics	$\frac{\frac{\alpha}{t_i^{1-\theta}}}{x[\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{x}{x_0}+1]^{\frac{2-\theta}{1-\theta}}}$	$\frac{\alpha}{\alpha(1-\theta)x\ln\frac{x}{x_0}+t_i^{1-\theta}x}$	$\frac{1}{1-\theta}\ln[\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{x}{x_0} + 1]$	$x_0 e^{\frac{t_i^{1-\theta}[e^{(1-\theta)y}-1]}{\alpha(1-\theta)}}$	$x_0 e^{\frac{t^{1-\theta}-t_i^{1-\theta}}{\alpha(1-\theta)}}$	$\frac{x_i(t)}{\alpha t^{\theta}}$	PA+ Non-linear GC
Generated by Sigmoid Dynamics*	$\frac{d}{dx}(\frac{\frac{\alpha}{Nt_i}\ln A}{1+\frac{\alpha}{Nt_i}\ln A})$	$\frac{d}{dx}(\ln[\frac{\alpha}{Nt_i}\ln A + 1])$	$\ln[\frac{\alpha}{Nt_i}\ln A + 1]$	$N\frac{Be^{\frac{Nt_i(e^y-1)}{\alpha}}}{1+Be^{\frac{Nt_i(e^y-1)}{\alpha}}}$	$N\frac{Be^{\frac{N(t-t_i)}{\alpha}}}{1+Be^{\frac{N(t-t_i)}{\alpha}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha}$	PA + EL
Generated by Log Logistic Dynamics*	$\frac{\alpha(\frac{N-x_0}{x_0})^{\frac{-\alpha}{N}} x^{-\frac{\alpha}{N}-1}}{N(N-x)^{\frac{-\alpha}{N}+1}}$	$\frac{\frac{\alpha}{N}}{x(N-x)}$	$\frac{\alpha}{N}\ln A$	$\frac{NBe^{\frac{N}{\alpha}y}}{1+Be^{\frac{N}{\alpha}y}}$	$N\frac{B(\frac{t}{t_i})^{\frac{N}{\alpha}}}{1+B(\frac{t}{t_i})^{\frac{N}{\alpha}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha t}$	PA + EL + GC
Generated by stretched Logistic Dynamics*	$-\frac{d}{dx}[1 + \frac{\alpha(1-\theta)}{Nt_i^{1-\theta}}\ln A]^{\frac{-1}{1-\theta}}$	$\frac{d}{dx}\frac{\ln[1+\frac{\alpha(1-\theta)}{Nt_i^{1-\theta}}\ln A]}{1-\theta}$	$\frac{\ln[1+\frac{\alpha(1-\theta)}{Nt_i^{1-\theta}}\ln A]}{1-\theta}$	$\frac{NBe^{\frac{N}{\alpha}\frac{t_i^{1-\theta}[e^{(1-\theta)y}-1]}{1-\theta}}}{1+Be^{\frac{N}{\alpha}\frac{t_i^{1-\theta}[e^{(1-\theta)y}-1]}{1-\theta}}}$	$\frac{NBe^{\frac{N}{\alpha}\frac{t^{1-\theta}-t_i^{1-\theta}}{1-\theta}}}{1+Be^{\frac{N}{\alpha}\frac{t^{1-\theta}-t_i^{1-\theta}}{1-\theta}}}$	$\frac{x_i(t)[N-x_i(t)]}{\alpha t^{\theta}}$	PA + EL + Non-linear GC
Generated by Confined Exponential	$\frac{\frac{\alpha}{t_i}\frac{1}{N-x}}{(1-\frac{\alpha}{t_i}\ln\frac{N-x}{N-x_0})^2}$	$\frac{\frac{\alpha}{t_i}\frac{1}{N-x}}{1-\frac{\alpha}{t_i}\ln\frac{N-x}{N-x_0}}$	$\ln[1 - \frac{\alpha}{t_i}\ln\frac{N-x}{N-x_0}]$	$N - \frac{N-x_0}{e^{\frac{t_i(e^y-1)}{\alpha}}}$	$N - \frac{N-x_0}{e^{\frac{(t-t_i)}{\alpha}}}$	$\frac{N-x_i(t)}{\alpha}$	EL
Generated by Confined Power law	$\frac{\alpha(N-x_0)^{-\alpha}}{(N-x)^{1-\alpha}}$	$\frac{\alpha}{N-x}$	$-\alpha\ln\frac{N-x}{N-x_0}$	$N - (N-x_0)e^{\frac{-y}{\alpha}}$	$N - (N-x_0)(\frac{t}{t_i})^{\frac{-1}{\alpha}}$	$\frac{N-x_i(t)}{\alpha t}$	GC + EL
Generated by Confined Stretched Exponentail	$\frac{\frac{\alpha}{t_i^{1-\theta}(N-x)}}{[1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{N-x}{N-x_0}]^{\frac{2-\theta}{1-\theta}}}$	$\frac{\frac{\alpha}{t_i^{1-\theta}(N-x)}}{1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{N-x}{N-x_0}}$	$\frac{\ln[1-\frac{\alpha(1-\theta)}{t_i^{1-\theta}}\ln\frac{N-x}{N-x_0}]}{1-\theta}$	$N - \frac{(N-x_0)}{e^{\frac{t_i^{1-\theta}(e^{y(1-\theta)}-1)}{\alpha(1-\theta)}}}$	$N - \frac{(N-x_0)}{e^{\frac{t^{1-\theta}-t_i^{1-\theta}}{\alpha(1-\theta)}}}$	$\frac{N-x_i(t)}{\alpha t^{\theta}}$	Non-linear GC + EL
Generated by Linear Dynamics	$\frac{\frac{\alpha}{t_i}}{[\frac{\alpha}{t_i}(x-x_0)+1]^2}$	$\frac{1}{(x-x_0)+\frac{t_i}{\alpha}}$	$\ln[\frac{\alpha}{t_i}(x - x_0) + 1]$	$x_0 + \frac{t_i}{\alpha}(e^y - 1)$	$x_0 + \frac{t-t_i}{\alpha}$	$\frac{1}{\alpha}$	Constant rate

Notes: t_i in the distribution means that the generated distribution is time-varing as dynamical system evolves. * $z = 1 - \frac{t_i}{t}$; $A = \frac{N-x_0}{x_0}\frac{x}{N-x}$, and $B = \frac{x_0}{N-x_0}$.